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NUMERICAL SOLUTION OF VARIABLE ORDER INTEGRO-DIFFERENTIAL EQUATIONS

R.M. Ganji[1], **H. Jafari**[1,2*]

[1]Department of Mathematics, University of Mazandaran, Babolsar, Iran
[2]Department of Mathematical Sciences, University of South Africa, UNISA 0003, South Africa

Abstract. In this paper, we use Taylor, Chebychev and Legendre polynomials as a basis polynomials to obtain numerical solution of the variable order integro-differential equations (VOIDEs). With the help the basis polynomials and collocation method, the VOIDEs are reduced to a system of algebraic equations. Then, we solve the system and obtain the approximate solution. Two examples are given to verify the efficiency of the proposed method.

Keywords: Variable order, integro-differential equations, the basis polynomials, collocation method.
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Corresponding author: Hossein Jafari, Department of Mathematical Sciences, University of South Africa, UNISA 0003, South Africa, e-mail: *jafari.usern@gmail.com*
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1 Introduction

Fractional order differential equations are successfully applied in physics and engineering such as earthquake analysis, bio-chemical, electric circuits, controller design, signal processing, viscoelasticity and so on Atanackovic et al. (2014); Chen et al. (2015); Podlubny (1999).

It is well know that obtaining of exact solution for most fractional ordinary/ partial/ integro-differential equations are difficult or even impossible. So numerical or even approximation schemes must be used. In last decades, some researches have proposed several approximation and numerical methods. For example, Legendre, Bernstein, Bernoulli and Chebyshev polynomials Heydari et al. (2014, 2017); Mohammadi & Hosseini (2011); Atanackovic et al. (2014); Yousefi & Behroozifar (2010); Rahimkhani et al. (2017); Bhrawy et al. (2013); Zhu & Fan (2012), Adomian decomposition method, He's variational iteration, homotopy perturbation transform methods Jafari & Daftardar-Gejji (2006); Jafari et al. (2013 a,b), sinh-Gordon equation expansion method (ShGEEM) Sulaiman et al. (2018), the cancer treatment model Ali Dokuyucu et al. (2018) and so on.

Recently, the concept of variable order calculus is taken into consideration. The Variable order derivative is proposed by Samko Samko & Ross (1993) in 1993. Several techniques proposed for handling numerical calculation of both variable order ordinary and integro-differential equations Chen et al. (2015); Jia et al. (2017); Liu et al. (2016); Lorenzo & Hartley (2002); Xu & Suat Ertürk (2014); Yi et al. (2013).

The aim of our work is to obtain numerical solution of VOIDEs using such common basis polynomials. We study the following type of the VOIDEs:

$$\begin{cases} {}_0D_t^{\eta(t)}\Theta(t) = \lambda_1 \int_0^t N_1(t,\xi)\ \Theta(\xi)\ d\xi + \lambda_2 \int_0^1 N_2(t,\xi)\ \Theta(\xi)\ d\xi + \gamma(t), \\ \Theta(0) = \Theta_0, \end{cases} \tag{1}$$

where $\eta(t)$ is bounded function in $[0,1]$ and $N_1(t,\xi), N_2(t,\xi)$ and $\gamma(t)$ are the known functions, whereas $\Theta(t)$ is unknown. Here ${}_0D_t^{\eta(t)}$ is variable order Caputo derivative which is defined below Chen et al. (2015); Samko & Ross (1993); Shen et al. (2012); Xu & Suat Ertürk (2014); Zhuang et al. (2009).

Definition 1. *The variable order Caputo derivative for* $0 < \eta(t) \leq 1$ *is defined as:*

$$ {}_0D_t^{\eta(t)}\Theta(t) = \begin{cases} \frac{1}{\Gamma(1-\eta(t))}\int_0^t (t-s)^{-\eta(t)}\Theta'(s)\, d\xi, & 0 < \eta(t) < 1, \\ \Theta'(t), & \eta(t) = 1. \end{cases} \tag{2} $$

It is easy to report the following result, namely

$$ {}_0D_t^{\eta(t)} t^m = \begin{cases} \dfrac{\Gamma(m+1)}{\Gamma(m-\eta(t)+1)} t^{m-\eta(t)}, & m \in \mathbb{N}, \\ 0, & m = 0. \end{cases} \tag{3} $$

2 The Method

2.1 Few basis polynomials

In this subsection, we briefly review the Taylor, Legendre and Chebyshev polynomials as basis polynomials.

(i) The Taylor basis polynomials of degree n are defined by:

$$ B_n(t) = t^n, \quad n = 1, 2, \ldots. \tag{4} $$

(ii) The shifted Legendre basis polynomials of degree n given by:

$$ L_0(t) = 1,\ L_1(t) = 2t - 1, $$
$$ L_n(t) = \sum_{k=0}^{n} (-1)^{n+k} \frac{(n+k)!\, t^k}{(n-k)!\,(k!)^2}, \quad n = 1, 2, \ldots. \tag{5} $$

(iii) The shifted Chebyshev polynomials $C_n^*(t)$ are defined in terms of the Chebyshev polynomials $C_n(t)$ by the following relation:

$$ C_n^*(t) = C_n(2t-1), \quad n = 1, 2, \ldots, \tag{6} $$

where $C_0(t) = 1, C_1(t) = 2t$ and $C_n(t) = 2t\, C_{n-1}(t) - C_{n-2}(t)$.

Let $\{P_0^*(t), P_1^*(t), ..., P_n^*(t)\} \subset H$, where $H = L^2[0,1]$ is a Hilbert space, be the set of one of the above polynomials. Let $S_n = Span\{P_0^*(t), P_1^*(t), ..., P_n^*(t)\}$ and $\Theta \in H$ be an arbitrary element. S_n is a complete subset of H because of S_n is a closed and finite dimensional subspace. So, Θ has the unique approximation out of S_n such as $\tilde{\Theta} \in S_n$. Therefore, exist the unique coefficients a_i $(i = 0, 1, ..., n)$ so that

$$ \Theta(t) \approx \tilde{\Theta}_n(t) = \sum_{i=0}^{n} a_i P_i^*(t), \tag{7} $$

where the function $\tilde{\Theta}_n(t)$ in the above equation is an approximate solution for Eq. (1).

2.2 Function approximation for Eq. (1)

Let $\tilde{\Theta}_n(t)$ in (7) be approximation of $\Theta(t)$, then by substituting (7) in Eq. (1) we have:

$$\begin{cases} {}_0D_t^{\eta(t)}\tilde{\Theta}_n(t) = \lambda_1 \int_0^t N_1(t,\xi)\ \tilde{\Theta}_n(\xi)\ d\xi + \lambda_2 \int_0^1 N_2(t,\xi)\ \tilde{\Theta}_n(\xi)\ d\xi + \gamma(t), \\ \tilde{\Theta}_n(0) = \Theta_0. \end{cases} \tag{8}$$

We define the residual function:

$$\begin{aligned} R(t, a_0, a_1, \ldots, a_n) &= {}_0D_t^{\eta(t)}\tilde{\Theta}_n(\xi) - \\ -\lambda_1 \int_0^t N_1(t,\xi)\ \tilde{\Theta}_n(\xi)\ d\xi &- \lambda_2 \int_0^1 N_2(t,\xi)\ \tilde{\Theta}_n(\xi)\ d\xi - \gamma(t) = 0. \end{aligned} \tag{9}$$

To find solution $\Theta(t)$, we use of the initial conditions in Eq. (8) and the roots of the shifted second kind of Chebyshev polynomial as the collocation point for obtain unknown coefficients $a_0, a_1, ..., a_n$. By substituting the collocation point in Eq. (9), we get the system of algebraic equations. Consequently $\Theta(t)$ given in (7) can be calculated.

3 Test Examples

In this section we solve two examples which is solved by Operational Matrix Method in Yi et al. (2013).

Example 1. *Yi et al. (2013)*

$$\begin{cases} {}_0D_t^{\eta(t)}\Theta(t) = \dfrac{1}{10} \int_0^t t\xi\ \Theta(\xi)\ d\xi + \dfrac{1}{3} \int_0^1 (t+\xi)\ \Theta(\xi)\ d\xi + \gamma(t), \\ \Theta(0) = 0, \end{cases}$$

where $0 \leq t \leq 1$, $\eta(t) = \dfrac{t}{2}$ *and*

$$\gamma(t) = \frac{\Gamma(7)t^{6-\frac{t}{2}}}{\Gamma(7-\frac{t}{2})} + \frac{\Gamma(8)t^{7-\frac{t}{2}}}{\Gamma(8-\frac{t}{2})} - \frac{t^9}{80} - \frac{t^{10}}{90} - \frac{5t}{56} - \frac{17}{216}.$$

The exact solution is $\Theta(t) = t^6 + t^7$. *By applying the proposed method for this example, the exact and approximation solutions are shown in figure 1. Also the absolute error are listed in the table 1:*

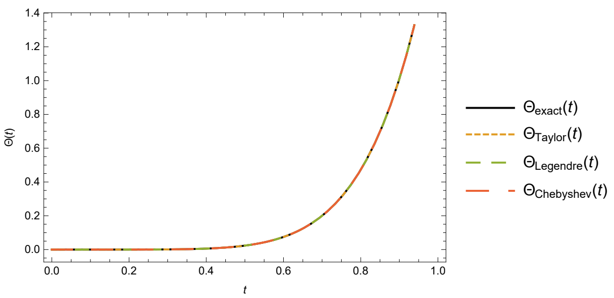

Figure 1: The exact and approximation solutions ($n = 7$)

Example 2. *Yi et al. (2013)*

$$\begin{cases} {}_0D_t^{\eta(t)}\Theta(t) = \int_0^t (t-\xi)\ \Theta(\xi)\ d\xi + \int_0^1 \xi \sin t\ \Theta(\xi)\ d\xi + \gamma(t), \\ \Theta(0) = 0, \end{cases}$$

Table 1: Absolute errors from variable basis polynomials ($n = 7$).

t	Error(Taylor)	Error(Chebychev)	Error(Legendre)	Error(Yi et al. (2013))
0	0	$6.93889e-18$	$3.33067e-16$	0
0.1	$2.31971e-17$	$8.97059e-17$	$8.12269e-16$	$2.107654e-006$
0.2	$7.74120e-17$	$2.99538e-16$	$3.33745e-16$	$7.584658e-007$
0.3	$2.13588e-17$	$1.92663e-16$	$6.74265e-16$	$5.452959e-006$
0.4	$1.04951e-16$	$2.43729e-16$	$8.59555e-16$	$5.800553e-006$
0.5	$6.93889e-17$	$5.75928e-16$	$3.36536e-16$	$3.502632e-006$
0.6	$2.77556e-17$	$5.96745e-16$	$3.05311e-16$	$1.525140e-004$
0.7	$5.55112e-17$	$4.71845e-16$	$2.77556e-17$	$5.296317e-004$
0.8	$1.66533e-16$	$6.10623e-15$	$1.38778e-15$	$1.558219e-003$
0.9	0	$1.33227e-15$	$2.44249e-15$	$4.035161e-002$

where $0 \leq t \leq 1$, $\eta(t) = t$ *and*

$$\gamma(t) = \frac{\Gamma(\frac{23}{4})}{\Gamma(\frac{23}{4}-t)} t^{\frac{19}{4}-t} + \frac{\Gamma(\frac{36}{5})}{\Gamma(\frac{36}{5}-t)} t^{\frac{31}{5}-t} - \frac{16}{621} t^{\frac{27}{4}} - \frac{25}{1476} t^{\frac{41}{5}} - \frac{299}{1107} sin(t).$$

By applying the proposed method for this example, we obtained numerical appropriate result. The exact solution $\Theta(t) = t^{\frac{19}{4}} + t^{\frac{31}{5}}$ *Yi et al. (2013), and approximation solution are ploted in figure 2. The absolute errors are presented in table 2:*

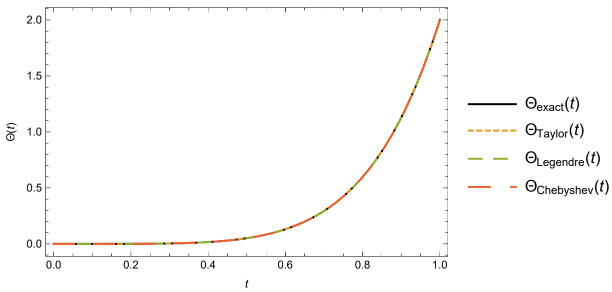

Figure 2: The exact and approximation solutions ($n = 7$)

Table 2: Absolute errors from variable basis polynomials ($n = 7$).

t	Error(Taylor)	Error(Chebychev)	Error(Legendre)	Error(Yi et al. (2013))
0	0	$1.11022e-16$	$4.71845e-16$	0
0.1	$4.39294e-6$	$4.39294e-7$	$4.39294e-7$	$8.246755e-004$
0.2	$2.69985e-7$	$2.69985e-6$	$2.69985e-6$	$8.785142e-006$
0.3	$5.5402e-6$	$5.5402e-7$	$5.5402e-7$	$9.112653e-005$
0.4	$4.11715e-6$	$4.11715e-6$	$4.11715e-6$	$6.645263e-005$
0.5	$3.40927e-6$	$3.4092e-6$	$3.4092e-6$	$1.256150e-003$
0.6	$2.27281e-6$	$2.27281e-6$	$2.27281e-6$	$9.275812e-003$
0.7	$4.77903e-7$	$4.77903e-7$	$4.77903e-6$	$9.578623e-003$
0.8	$1.72017e-6$	$1.72017e-6$	$1.72017e-7$	$6.632365e-003$
0.9	$2.71873e-7$	$2.71873e-7$	$2.71873e-6$	$8.658236e-002$

4 Conclusion

In this paper, we obtain numerical solution for a type of Integro-differential equations which has variable order derivative. Since the kernel of variable order derivative is very complex for having a variable exponent, it is not simply task to obtain the solution of such equations. Therefore developing an effective numerical algorithms for solving such equations is importance. We used few basis polynomials and the collocation method to obtain the approximate solution. The scheme is easy and efficient. It could be applied for other type of integro-differential equations.

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